Some Elementary Variational Solutions
OF PHYSICS AND ENGINEERING

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1 Introduction

Variational problems are problems that are cast into and solved using techniques from the calculus of variations. Variational techniques are important and powerful tools in physics, and they help explain many physical phenomena. Such solutions are also important in various other branches of sciences and engineering.

In this short note we deal with variationsl problems which are elementary in nature. Many more complex problems other than those discussed here exist. Variational methods is a large subject and we present some of the simplest starting solutions.

2 Basic Equations

The calculus of variations was developed to solve problems in which integrals of functions are to be minimized. Such integrals occur in physics, mathematics, and engineering. The minimizing solution to such integrals often explains physical phenomenon.

In one dimension this problem is described by finding the function $y(x)$ so that the integral of the function $F(x, y, y')$ is minimized. The function F depends on the variables x, y, and y' which are treated as independent variables. Here y' is the derivative

$$y' = \frac{dy(x)}{dx} \ .$$

That is the function $y(x)$ is to be found that extremizes the integral

$$J[y] = \int_{x_1}^{x_2} F(x, y, y')dx \ .$$

The endpoint values of y, $y(x_1)$ and $y(x_2)$, are considered to be fixed. That is, $y(x_1) = a$ and $y(x_2) = b$.

This optimization problem differs from an ordinary minimization or maximization problem in calculus in that it is a *functional* and not a function is being minimized. Whereas minimizing a function such as $f(x)$ involves finding a variable x, usually a real number, that minimizes $f(x)$, the current variational problem is different. It consists of finding a *function* y such that the *functional* $J[y]$ is minimized. In the first case the number to minimize depends on a real variable x and in the second and more difficult case the number to minimize depends on a function y.

The above variational minimization problem was solved over a century ago. Its solution is given by solving the so-called Euler Lagrange equation for this problem

$$\frac{\partial F(x,y,y')}{\partial y} - \frac{d}{dx}\left(\frac{\partial F(x,y,y')}{\partial y'}\right) = 0$$

subject to the constraints $y(x_1) = a$ and $y(x_2) = b$.

The function $F(x,y,y')$ is differentiated with respect to the second argument y and with respect to the third argument $y' = y'(x)$, and this latter derivative is then differentiated with respect to x once more.

More complex variational problems involve several variables $y_1, y_2, \ldots, y_n$ and their derivatives $y'_1, y'_2, \ldots, y'_n$. Still more complex problems involving minimizing (or maximizing) one integral while enforcing one more more other integral equality constraints. The additional constraints can also involve differential equations such as for advanced control systems optimization. These more advanced problems are not discussed further in this note.

Some applications of the above solution are given for some example problems.

3 The Brachistochrome Problem

The brachistochrome problem[1] consists of finding a curve so that if a ball were to roll down the curve under the force (acceleration) of gravity, it would roll down the curve in minimum time. The problem is to find the curve that minimizes the time to traverse from a to b. This problem turns out to be that of minimizing the integral

$$\int_a^b \sqrt{\frac{1+y'^2}{y}}\,dx$$

where $F(x,y,y') = \sqrt{\frac{1+y'^2}{y}}$ and where $y(x)$ describes the curve that the ball rolls down. The function $y(x)$ is to be found such that the integral is minimized.

It has been implicitly assumed in the derivations that $y(0) = 0$.

Calculation of some derivatives gives

$$\frac{\partial F(x,y,y')}{\partial y} = \frac{-\left(1+y'(x)^2\right)}{2\,y(x)^2\sqrt{\frac{1+y'(x)^2}{y(x)}}}$$

and

[1] This problem is around 300 years old.

$$\frac{\partial F(x,y,y')}{\partial y'} = \frac{y'(x)}{y(x)\sqrt{\frac{1+y'(x)^2}{y(x)}}}$$

so that

$$\frac{d}{dx}\left(\frac{\partial F(x,y,y')}{\partial y'}\right) = \frac{-y'(x)^2 - y'(x)^4 + 2\,y(x)\,y''(x)}{2\,y(x)^3\left(\frac{1+y'(x)^2}{y(x)}\right)^{\frac{3}{2}}} \; .$$

Combining the previous equations gives the Euler-Lagrange equation as

$$\frac{-1 - y'(x)^2 - 2\,y(x)\,y''(x)}{2\,y(x)^3\left(\frac{1+y'(x)^2}{y(x)}\right)^{\frac{3}{2}}} = 0 \; .$$

Taking the numerator to be zero gives

$$-1 - y'(x)^2 - 2\,y(x)\,y''(x) = 0$$

as the differential equation to be solved subject to the initial conditions.

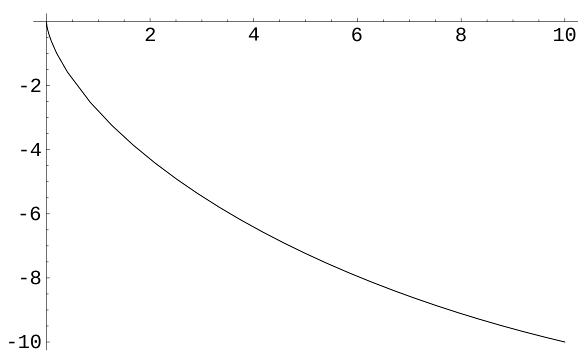
4 Numeral Solution

This variational equation is difficult to solve numerically because of a singularity at the origin. A trick to avoid this is to reverse and "flip" the solution by performing the substitution

$$z(x) = -y(a - x)$$

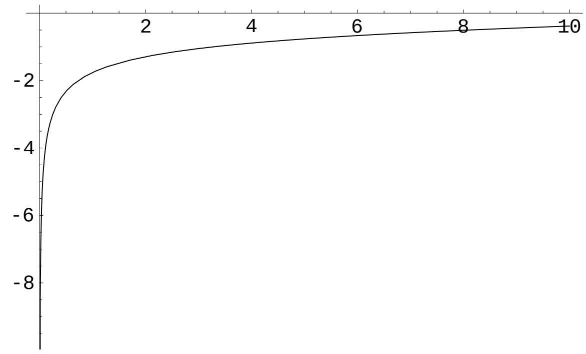
where for the solution considered here $a = 10$. It is easily found that $z(x)$ and $y(x)$ satisfy the same differential equation. This then allows the end condition to be used as the initial condition and then the singular point (appearing as the final point in z) is "gently" approached.

We illustrate a numerical solution for the time interval $[0, 10]$ with a final value of $y(10) = -10$.



$y(t)$ plotted over $[0, 10]$ is shown above.

This problem is difficult numerically because the origin $(0, 0)$ is singular. The derivative is infinite at this point. A numerically calculated derivative is shown below.



The solution above appears intuitive. The ball begin rolling down the slope at an infinite slope to gain speed as fast as possible, and then the curvature gently decreases as the ball rolls to the final point $[10, -10]$ in minimum time.

A good problem for the numerically inclined reader is to attempt the same problem with the initial condition $y(10) = -0.1$.

5 Closed Form Solution

It is verified by direct calculation that the closed form parameterized solution is given by

$$\begin{aligned} x(\theta) &= k(\theta - \sin(\theta)) \\ y(\theta) &= k(1 - \cos(\theta)) \end{aligned}$$

where k is a constant. This is the equation of a cycloid.

This classical solution is well known.