Derivations of Fibonacci Related Numbers

FEREGO research

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Contents

1 Introduction

This short note gives a derivation of a set of numbers similar to the Fibonacci numbers. There numbers will be called the Bob numbers or the Bob integers.[2] These numbers are defined more carefully below. The technique of matrix analysis using matrix diagonalization is used to find a closed form expression for these numbers.

The author has previously written two papers[3] around the time frame of 1990 which contain in essence the derivations given here.

It is also shown that powers of the so-called golden ratio $\psi = \frac{1+\sqrt{5}}{2}$

[1]George Schils is no longer at FEREGO. Contact him via the web or email.

[2]These are named after Robert Bastaz whom performed some derivations of this result. This derivation is not presented here however.

[3]These papers were written by the author, and are lost and missing.

$$\psi^n = \left(\frac{1+\sqrt{5}}{2}\right)^n$$

approach an exact integer as n gets large.

2 Definition of Bob numbers

The so-called Bob integers are defined as the sequence of numbers 1, 3, 4, 7, 11, and so on. The next number is obtained as the sum of the previous two numbers. If the n^{th} number is defined as $x(n)$ then this relation is described by

$$x(n+2) = x(n+1) + x(n) \ , \tag{1}$$

with

$$x(0) = 1, \ \ x(1) = 3 \ . \tag{2}$$

This equation is solved in closed form in the next section.

3 Derivation of n^{th} sequence term

Equation 1 is a difference equation. Such equations are well studied in mathematics and many well known solution techniques exist. Here we will use the technique of matrices to express and then solve the system equation given by Eq. 1.

To cast Eq. 1 into the form of a matrix system, first define the vector $v(n)$ as

$$v(n) = \begin{bmatrix} x(n+1) \\ x(n) \end{bmatrix} \ . \tag{3}$$

Then Eq. 1 can be expressed matrix form as

$$v(n+1) = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} v(n) \ , \tag{4}$$

as can be seen using the rules of matrix multiplication.

Defining the system matrix A for this problem as

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \tag{5}$$

then Eq. 4 assumes the simpler form

$$v(n+1) = A\,v(n)\ , \tag{6}$$

where multiplication means matrix multiplication.

This equation is to be solved subject to the initial condition

$$v(0) = \begin{bmatrix} 3 \\ 1 \end{bmatrix}\ . \tag{7}$$

It is easy to verify by induction that the solution to Eq. 6 for any A is given by

$$v(n) = A^n v(0)\ , \tag{8}$$

where A^n denotes the matrix A raised to the n^{th} power.

If the matrix A is diagonizable via the similarity transform S matrix, then A can be expressed as

$$A = S \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} S^{-1}\ . \tag{9}$$

Not every matrix A is expressible as given above, but for this analysis the existence of the representation given by Eq. 9 is assumed.

The n^{th} power of such a diagonizable matrix is then easily expressed as

$$A^n = S \begin{bmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{bmatrix} S^{-1}\ , \tag{10}$$

which is also easily obtained by induction on n.

We have actually solved a general 2×2 system, assuming diagonizability. An $n \times n$ is solved similarly.

From Eq. 8 the general solution of the system is

$$v(n) = S \begin{bmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{bmatrix} S^{-1}\, v(0)\ . \tag{11}$$

This solution is for any A that is diagonizable with arbitrary initial condition vector $v(0)$. It is noted that Eq. 11 holds for a general $n \times n$ system.

4 Specific Solution

Our interest in this paper is to solve the system equations 6 and 7 for the specific matrix given by Eq. 5.

We can verify by direct multiplication the relation

$$\frac{1}{\sqrt{5}}\begin{bmatrix} 1 & 1 \\ -\frac{\sqrt{5}+1}{2} & \frac{\sqrt{5}-1}{2} \end{bmatrix}\begin{bmatrix} \frac{1-\sqrt{5}}{2} & 0 \\ 0 & \frac{1+\sqrt{5}}{2} \end{bmatrix}\begin{bmatrix} \frac{\sqrt{5}-1}{2} & -1 \\ \frac{\sqrt{5}+1}{2} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \tag{12}$$

and so comparing to Eq. 9

$$\lambda_1 = \frac{1-\sqrt{5}}{2} \tag{13}$$

$$\psi = \lambda_2 = \frac{1+\sqrt{5}}{2} \tag{14}$$

and

$$S = \begin{bmatrix} 1 & 1 \\ -\frac{\sqrt{5}+1}{2} & \frac{\sqrt{5}-1}{2} \end{bmatrix}. \tag{15}$$

It is clear that $\lambda_1\lambda_2 = -1$, so the λ's are negative reciprocals. The number ψ is called the *golden ratio.*

Multiplying out Eq. 11, making substitutions from Eqs. 13-15, performing some algebra, and taking the second element of the vector gives the desired representation for the n^{th} Bob integer

$$x(n) = \left(\frac{1+\sqrt{5}}{2}\right)^{n+1} + \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}. \tag{16}$$

This is a remarkable result!

5 Integer properties of λ_1^n

It is clear that $x(n)$ is always an integer for all n. A simple manipulation of Eq. 16 gives

$$\left(\frac{1+\sqrt{5}}{2}\right)^{n} = x(n-1) - \left(\frac{1-\sqrt{5}}{2}\right)^{n}. \tag{17}$$

The second term on the right hand side of this equation becomes exponentially small as n increases, and the first term on the right hand side is always an integer. Thus the left hand side of Eq. 17 exponentially approaches an exact integer as n increases.

For example, the 149^{th} Bob integer is 22291846172619859445381409012498 and the numerically calculated value of λ_1^{150} is

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22291846172619859445381409012497.99999999999999999999999999999999955140548
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6 Generalized Bob Number Representations

A critical assumption in the above analysis and the assumption that results in the $x(n)$ being integers is the assumption on the initial conditions. In the previous section, the initial condition values were 1 and 3. In this section, the analysis of the previous section is generalized to the case where the starting values are any arbitrary integers n_1 and n_2.

$$v(0) = \begin{bmatrix} n_2 \\ n_1 \end{bmatrix} . \tag{18}$$

Evaluating Eq. 11 and repeating the above analysis produces the generalized representation

$$x(n) = \left(\frac{1+\sqrt{5}}{2}\right)^n \left[n_1\left(\frac{5-\sqrt{5}}{10}\right) + \frac{n_2}{\sqrt{5}}\right] + \left(\frac{1-\sqrt{5}}{2}\right)^n \left[n_1\left(\frac{5+\sqrt{5}}{10}\right) - \frac{n_2}{\sqrt{5}}\right] . \tag{19}$$

Performing steps similar to those associated with Eq. 17 and noting that $x(n)$ is integer for all n shows that the quantity

$$\left(\frac{1+\sqrt{5}}{2}\right)^n \left[n_1\left(\frac{5-\sqrt{5}}{10}\right) + \frac{n_2}{\sqrt{5}}\right] \tag{20}$$

exponentially approaches an integer as n gets large. This is true for all integer n_1 and n_2.

A special case of Eq. 19 is when $n_1 = 0$ and $n_2 = 1$, resulting in the well known Fibonacci sequence $F_0 = 0,\ F_1 = 1,\ F_2 = 1,\ F_3 = 2,\ F_4 = 3,\ F_5 = 5,\ \ldots$. Using the notation F_n to denote the n^{th} Fibonacci number, Eq. 19 becomes

$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right] . \tag{21}$$

This is also an elegant form.

Finally, the F_n symbols can be used to express the generalized $x(n)$ numbers of Eq. 19. First we note that the numbers begin as $x(0) = n_1,\ x(1) = n_2,\ x(2) = n_1 + n_2,\ x(3) = n_1 + 2n_2,\ x(4) =$

$2n_1 + 3n_2$, ... and it is seen that the coefficients of the n_1 and n_2 are the Fibonacci numbers F_n themselves. This relation is stated as

$$x(n+1) = F_n \, n_1 + F_{n+1} \, n_2 \; . \tag{22}$$

This is easily proved by induction on n using the Fibonacci relation $F_{n+2} = F_{n+1} + F_n$ as well as Eq. 1. This result alternatively follows by using Eq. 21 to give the coefficients in Eq. 19.

7 Summary

The technique of matrix analysis has been used to solve for a closed form expression for a set of numbers related to the Fibonacci numbers. Both the end result as well as the derivation technique are interesting. A remarkable property regarding golden ratio powers has been proved and an example has been given. Results were generalized to include all starting seeds for Fibonacci type sequences, and a more general golden ratio result was shown.