An Introduction to Ramamurthi Fraction Identities

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1 Introduction

The number theoretic results found by Ramanujan are some of the most obscure and most interesting results in mathematics. Ramanujan would produce interesting and amazing results, often seemingly by "pulling them out of a hat".

In this short paper we explore some fraction power identities which are similar in nature to some of Ramanujan's results. In particular, we explore a set of identities which are somewhat similar in form to the so-called "third notebook" identity.

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[1]Work done at FEREGO Research. No longer at FEREGO Research.

[2]See http://nivedh.com/maths/1729-SOP/1729-SOPtext.htm.

2 Third Notebook Identity

We state the "third notebook" result of Ramunujan not because it is directly useful to this work but because it serves as motivation for the work presented here. The form of this identity is similar to the results presented later in this paper. The third notebook (TB)[3] result states that if F_n is defined as

$$F_n(a,b,c,d) = (a+b+c)^n + (b+c+d)^n - (c+d+a)^n - (d+a+b)^n + (a-d)^n - (b-c)^n$$

and if $a\,d = b\,c$ then

$$64\ F_6(a,b,c,d)\ F_{10}(a,b,c,d) = 45\ F_8^2(a,b,c,d)\ .$$

It is possible to prove this formula using modern symbolic algebra systems. We have performed such a proof. Such symbolic proofs although verifying the result still leave some uncertainty in how the results comes about.

The results in this paper are also motivated by work performed by Rajesh Ram.[4] The current paper attempts to motivate, prove, and expand on the results given at this web reference.

3 Definitions and TB Motivation

Define the functions $P(m)$, $Q(m)$, $R(m)$, and $S(m)$ for complex a, b, and m as:

$$\begin{aligned}
P_1(m,a,b) = P(m) = &\quad (11a+6b)^m + (-5a-11b)^m + (-6a+5b)^m \\
Q_1(m,a,b) = Q(m) = &\quad (a-9b)^m + (-10a-b)^m + (9a+10b)^m \\
R_1(m,a,b) = R(m) = &\quad (6a+11b)^m + (-11a-5b)^m + (5a-6b)^m \\
S_1(m,a,b) = S(m) = &\quad (-9a+b)^m + (-a-10b)^m + (10a+9b)^m\ .
\end{aligned}$$

The dependence on a, b is suppressed for the functions $P(m)$, $Q(m)$, $R(m)$, and $S(m)$ whereas the functions with subscript "1" will be useful in cases where the dependence on a, b needs to be explicitly stated.

These formulas for $P(m)$, $Q(m)$, $R(m)$, and $S(m)$ are obscure, and some motivation for these forms is provided. These forms follow if certain substitutions are made, as is now done.

Consider the following transformations from a, b into "primed coordinates"[5] given by

$$\begin{aligned}
a_1' = &\quad 3a+b \\
b_1' = &\quad -4(3a+b) \\
c_1' = &\quad -1(2a+3b) \\
d_1' = &\quad 4(2a+3b)\ ,
\end{aligned}$$

[3]Berndt, Bruce C, Bhargava, S., A remarkable identity found in Ramanujan's third notebook, *Glascow Math. J.*, **34**, no. 3, 341-345.

[4]See the web address http://nivedh.com/maths/1729-SOP/1729-SOPtext.htm.

[5]These transformations were used by Ramamurthi (personal correspondence).

and

$$\begin{aligned} a_2' =& \quad 4(2a-b) \\ b_2' =& \quad -1(2a-b) \\ c_2' =& \quad -4(3a+2b) \\ d_2' =& \quad 3a+2b \; . \end{aligned}$$

Some algrebra, using the above substitutions, shows that the formulas for $P(m)$, $Q(m)$, $R(m)$, and $S(m)$ are given by

$$\begin{aligned} P_1(m,a,b) =& \quad (-a_1'-b_1'-c_1')^m + (b_1'+c_1'+d_1')^m + (a_1'-d_1')^m \\ Q_1(m,a,b) =& \quad (-a_1'-b_1'-d_1')^m + (a_1'+c_1'+d_1')^m + (b_1'-c_1')^m \\ R_1(m,a,b) =& \quad (-a_2'-b_2'-c_2')^m + (b_2'+c_2'+d_2')^m + (a_2'-d_2')^m \\ S_1(m,a,b) =& \quad (-a_2'-b_2'-d_2')^m + (a_2'+c_2'+d_2')^m + (b_2'-c_2')^m \; . \end{aligned}$$

It is assumed that $(-1)^m = 1$ or that m is even.

Simple inspection shows that the previous equations are related to the F_n formula as follows

$$\begin{aligned} F_n(a_1',b_1',c_1',d_1') =& \quad P_1(m,a,b) - Q_1(m,a,b) \\ F_n(a_2',b_2',c_2',d_2') =& \quad R_1(m,a,b) - S_1(m,a,b) \; . \end{aligned}$$

Because $a_1'd_1' - b_1'c_1' = 0$ and $a_2'd_2' - b_2'c_2' = 0$ the assumptions on the TB formula are satisfied. Applying the TB formula using the previous equations gives the "Ramanujan form" of the identities as

$$\begin{aligned} 64\,[P_1(6,a,b)-Q_1(6,a,b)]\,[P_1(10,a,b)-Q_1(10,a,b)] =& \quad 45\,[P_1(8,a,b)-Q_1(8,a,b)]^2 \\ 64\,[R_1(6,a,b)-S_1(6,a,b)]\,[R_1(10,a,b)-S_1(10,a,b)] =& \quad 45\,[R_1(8,a,b)-S_1(8,a,b)]^2 \; . \end{aligned}$$

Dividing the first equation above by the second gives the *weak form* of the Ramamurthi fraction result as

$$\left(\tfrac{P_1(6,a,b)-Q_1(6,a,b)}{R_1(6,a,b)-S_1(6,a,b)}\right)\left(\tfrac{P_1(10,a,b)-Q_1(10,a,b)}{R_1(10,a,b)-S_1(10,a,b)}\right) = \left(\tfrac{P_1(8,a,b)-Q_1(8,a,b)}{R_1(8,a,b)-S_1(8,a,b)}\right)^2 \; .$$

A number of useful results are obtained by various arrangements of the previous two equations. These are not listed here.

The next section will strengthen this result by showing that each term in parenthesis is indeed equal to the other terms for all a and b.

It is useful to define the linear fraction form

$$G_1(m,a,b) = \frac{P_1(m,a,b)-Q_1(m,a,b)}{R_1(m,a,b)-S_1(m,a,b)} \; .$$

The subscript "1" indicates that this is a *first order* fraction. This notation will be convenient for some future work.

The TB formula remains a mystery to mathematicians, even years after Ramanujan's death. A brief digression is made to provide some justification (or "intuition") of this result in the special primed coordinate systems. In the special case where the coordinates are transformed into the primed coordinate system, the TB formula follows by observing the factorizations[6]

$$\begin{aligned} F_6(a_1', b_1', c_1', d_1') =& \ 2520\ (a-2\,b)\ (4\,a-b)\ (3\,a+b)\ (2\,a+3\,b)\ (5\,a+4\,b)\ (a+5\,b) \\ F_{10}(a_1', b_1', c_1', d_1') =& \ 104340600\ (a-2\,b)\ (4\,a-b)\ (3\,a+b)\ (2\,a+3\,b)\ (5\,a+4\,b)\ (a+5\,b)\ \left(a^2+a\,b+b^2\right)^2 \\ F_8(a_1', b_1', c_1', d_1') =& \ 611520\ (a-2\,b)\ (4\,a-b)\ (3\,a+b)\ (2\,a+3\,b)\ (5\,a+4\,b)\ (a+5\,b)\ \left(a^2+a\,b+b^2\right)\ . \end{aligned}$$

Recall that these F_n formulas in primed coordinates are equal to the $P-Q$ and $R-S$ expressions as functions of a and b. This fact is used in the next section.

An easy multiplication shows that $64\,F_6\,F_{10} = 45\,F_8^2$ in the a_1', b_1', c_1', d_1' primed coordinate system.

The TB formula follows easily by observing the following factorizations in the second a_2', b_2', c_2', d_2' coordinate system

$$\begin{aligned} F_6(a_2', b_2', c_2', d_2') =& \ 2520\ (a-4\,b)\ (2\,a-b)\ (5\,a+b)\ (3\,a+2\,b)\ (a+3\,b)\ (4\,a+5\,b) \\ F_{10}(a_2', b_2', c_2', d_2') =& \ 104340600\ (a-4\,b)\ (2\,a-b)\ (5\,a+b)\ (3\,a+2\,b)\ (a+3\,b)\ (4\,a+5\,b)\ \left(a^2+a\,b+b^2\right)^2 \\ F_8(a_2', b_2', c_2', d_2') =& \ 611520\ (a-4\,b)\ (2\,a-b)\ (5\,a+b)\ (3\,a+2\,b)\ (a+3\,b)\ (4\,a+5\,b)\ \left(a^2+a\,b+b^2\right)\ . \end{aligned}$$

These above factorizations are deep and mysterious results. These factorizations result in an easy proof of the TB formula (for the special primed coordinates case), and they also give easy proofs of many of the fraction results of the next section.

4 Linear Fraction Identities

Values of m for which the above fraction $G_1(m, a, b)$ results in identities are of interest. This section explores some relationships among the $G_1(m, a, b)$. It is verified by algebra that

$$\begin{aligned} \tfrac{P(3)-Q(3)}{R(3)-S(3)} &= \tfrac{P(5)-Q(5)}{R(5)-S(5)} \\ &= \tfrac{P(7)-Q(7)}{R(7)-S(7)} \\ &= \tfrac{-6\,a^3+a^2\,b+19\,a\,b^2+6\,b^3}{6\,a^3+19\,a^2\,b+a\,b^2-6\,b^3} \end{aligned}$$

hold for all complex a and b not resulting in singularities. We call this the $\{3, 5, 7\}$ linear fraction result.

This result holds as an identity but is of a different nature than the next results presented.

We mention for completeness that the following trivial case exists

$$\begin{aligned} \tfrac{P(2)-Q(2)}{R(2)-S(2)} &= \tfrac{P(4)-Q(4)}{R(4)-S(4)} \\ &= 0\ . \end{aligned}$$

[6]A later paper shows these results in a more general coordinate system.

This $\{2,4\}$ linear fraction result appears uninteresting.

The most interesting result is

$$\begin{aligned}
\frac{P(6)-Q(6)}{R(6)-S(6)} &= \frac{P(8)-Q(8)}{R(8)-S(8)} \\
&= \frac{P(10)-Q(10)}{R(10)-S(10)} \\
&= \frac{120\,a^6+646\,a^5\,b-185\,a^4\,b^2-2400\,a^3\,b^3-1615\,a^2\,b^4+74\,a\,b^5+120\,b^6}{120\,a^6+74\,a^5\,b-1615\,a^4\,b^2-2400\,a^3\,b^3-185\,a^2\,b^4+646\,a\,b^5+120\,b^6} \\
&= \frac{(a-2\,b)\,(4\,a-b)\,(3\,a+b)\,(2\,a+3\,b)\,(5\,a+4\,b)\,(a+5\,b)}{(a-4\,b)\,(2\,a-b)\,(5\,a+b)\,(3\,a+2\,b)\,(a+3\,b)\,(4\,a+5\,b)}\ .
\end{aligned}$$

The last equation above follows from the above equations. In particular, it follows from the factorization of the $F_n, n = 6, 8, 10$ ($P-Q$ or $R-S$) in the primed coordinates. It is seen that the $m = 8$ form has the factor $\left(a^2+a\,b+b^2\right)$ in both the numerator and denominator; and it is seen that the $m = 10$ form has the factor $\left(a^2+a\,b+b^2\right)^2$ in both the numerator and denominator. These common factors for the $m = 8$ and $m = 10$ cases cancel, making each result equal to the $m = 6$ case.

This $\{6,8,10\}$ linear fraction result appears in the quadratic results of the next section.

Nice factorizations occur for the P, Q, R, S functions. These factorizations are mentioned for completeness. These factorizations show that

$$\begin{aligned}
P(6) &= 1833842\,a^6+5770806\,a^5\,b+9526515\,a^4\,b^2 \\
&\quad +8537420\,a^3\,b^3+8180115\,a^2\,b^4+5232246\,a\,b^5+1833842\,b^6 \\
Q(6) &= 1531442\,a^6+4142886\,a^5\,b+9992715\,a^4\,b^2 \\
&\quad +14585420\,a^3\,b^3+12249915\,a^2\,b^4+5045766\,a\,b^5+1531442\,b^6 \\
R(6) &= 1833842\,a^6+5232246\,a^5\,b+8180115\,a^4\,b^2 \\
&\quad +8537420\,a^3\,b^3+9526515\,a^2\,b^4+5770806\,a\,b^5+1833842\,b^6 \\
S(6) &= 1531442\,a^6+5045766\,a^5\,b+12249915\,a^4\,b^2 \\
&\quad +14585420\,a^3\,b^3+9992715\,a^2\,b^4+4142886\,a\,b^5+1531442\,b^6\ ,
\end{aligned}$$

$$\begin{aligned}
P(8) &= 182\,\left(a^2+a\,b+b^2\right)\left(1189171\,a^6+3926553\,a^5\,b+5166310\,a^4\,b^2\right. \\
&\quad \left.+2591565\,a^3\,b^3+3371110\,a^2\,b^4+3208473\,a\,b^5+1189171\,b^6\right) \\
Q(8) &= 182\,\left(a^2+a\,b+b^2\right)\left(785971\,a^6+1755993\,a^5\,b+5787910\,a^4\,b^2\right. \\
&\quad \left.+10655565\,a^3\,b^3+8797510\,a^2\,b^4+2959833\,a\,b^5+785971\,b^6\right) \\
R(8) &= 182\,\left(a^2+a\,b+b^2\right)\left(1189171\,a^6+3208473\,a^5\,b+3371110\,a^4\,b^2\right. \\
&\quad \left.+2591565\,a^3\,b^3+5166310\,a^2\,b^4+3926553\,a\,b^5+1189171\,b^6\right) \\
S(8) &= 182\,\left(a^2+a\,b+b^2\right)\left(785971\,a^6+2959833\,a^5\,b+8797510\,a^4\,b^2\right. \\
&\quad \left.+10655565\,a^3\,b^3+5787910\,a^2\,b^4+1755993\,a\,b^5+785971\,b^6\right)\ ,
\end{aligned}$$

and

$$\begin{aligned}
P(10) &= 8281\,(a^2+a\,b+b^2)^2\,(3140642\,a^6+10768326\,a^5\,b+11461167\,a^4\,b^2\\
&\quad +487124\,a^3\,b^3+4729167\,a^2\,b^4+8075526\,a\,b^5+3140642\,b^6)\\
Q(10) &= 8281\,(a^2+a\,b+b^2)^2\,(1628642\,a^6+2628726\,a^5\,b+13792167\,a^4\,b^2\\
&\quad +30727124\,a^3\,b^3+25078167\,a^2\,b^4+7143126\,a\,b^5+1628642\,b^6)\\
R(10) &= 8281\,(a^2+a\,b+b^2)^2\,(3140642\,a^6+8075526\,a^5\,b+4729167\,a^4\,b^2\\
&\quad +487124\,a^3\,b^3+11461167\,a^2\,b^4+10768326\,a\,b^5+3140642\,b^6)\\
S(10) &= 8281\,(a^2+a\,b+b^2)^2\,(1628642\,a^6+7143126\,a^5\,b+25078167\,a^4\,b^2\\
&\quad +30727124\,a^3\,b^3+13792167\,a^2\,b^4+2628726\,a\,b^5+1628642\,b^6)\;.
\end{aligned}$$

It is seen that the $m=8$ terms have the common factor $(a^2+a\,b+b^2)$ whereas the $m=10$ terms have the common factor $(a^2+a\,b+b^2)^2$. Therefore the above equation can also be used to demonstrate the $\{6,8,10\}$ result.

To obtain the next results, observe that both the $\{3,5,7\}$ linear fraction result as well as the $\{6,8,10\}$ linear fraction result have forms where the numerator and denominator are obtained from each other by interchanging the roles of a and b.

Expressing the dependendence on a and b via the functions P_1, Q_1, R_1, and S_1 this symmetry can be expressed as

$$\begin{aligned}
\frac{P_1(6,a,b)-Q_1(6,a,b)}{R_1(6,a,b)-S_1(6,a,b)} &= \frac{P_1(8,a,b)-Q_1(8,a,b)}{R_1(8,a,b)-S_1(8,a,b)}\\
&= \frac{P_1(10,a,b)-Q_1(10,a,b)}{R_1(10,a,b)-S_1(10,a,b)}\\
&= \frac{120\,a^6+646\,a^5\,b-185\,a^4\,b^2-2400\,a^3\,b^3-1615\,a^2\,b^4+74\,a\,b^5+120\,b^6}{120\,a^6+74\,a^5\,b-1615\,a^4\,b^2-2400\,a^3\,b^3-185\,a^2\,b^4+646\,a\,b^5+120\,b^6}\\
&= \frac{R_1(6,b,a)-S_1(6,b,a)}{P_1(6,b,a)-Q_1(6,b,a)}\\
&= \frac{R_1(8,b,a)-S_1(8,b,a)}{P_1(8,b,a)-Q_1(8,b,a)}\\
&= \frac{R_1(10,b,a)-S_1(10,b,a)}{P_1(10,b,a)-Q_1(10,b,a)}\;.
\end{aligned}$$

The last three equations above were obtained by interchanging the roles of a and b and taking reciprocals.

Cross multiplication from the last equation produces another series of results, of which a few are listed below

$$\begin{aligned}
(P_1(6,a,b)-Q_1(6,a,b))(P_1(6,b,a)-Q_1(6,b,a)) &=\\
(R_1(6,a,b)-S_1(6,a,b))(R_1(6,b,a)-S_1(6,b,a))\;,\\
(P_1(8,a,b)-Q_1(8,a,b))(P_1(8,b,a)-Q_1(8,b,a)) &=\\
(R_1(8,a,b)-S_1(8,a,b))(R_1(8,b,a)-S_1(8,b,a))\;,\\
(P_1(10,a,b)-Q_1(10,a,b))(P_1(10,b,a)-Q_1(10,b,a)) &=\\
(R_1(10,a,b)-S_1(10,a,b))(R_1(10,b,a)-S_1(10,b,a))\;,\\
(P_1(6,a,b)-Q_1(6,a,b))(P_1(10,b,a)-Q_1(10,b,a)) &=\\
(R_1(6,a,b)-S_1(6,a,b))(R_1(10,b,a)-S_1(10,b,a)) &\;.
\end{aligned}$$

5 Quadratic Fraction Identities

It is remarkable that the same forms appearing in the previous section also appear in more complex contexts.

A notation for dealing with quadratic Ramamurthi forms is introducted in this section.

A special notation is defined to express more general Ramamurthi fractions. First the linear notations is repeated with some extensions to denote the particular permutation of the set

$$Z = \{P, Q, R, S\}$$

that is used.

$$F_L(Z, n) = F_L(\{P, Q, R, S\}, n) = \frac{P(n) - Q(n)}{R(n) - S(n)} \,.$$

Here the dependence on P, Q, R, S is explicitly shown.

In order to consider different permutations of the set Z, define the i^{th} permutation of the set Z as

$$Z_i = \{P_i, Q_i, R_i, S_i\}.$$

Here the elements P_i, Q_i, R_i, S_i are any permutation of the set Z. Thus Z_i denotes any permutation; Z_j denotes another permutation, possibly the same one.

The general quadratic Ramamurthi fraction form is thus defined as

$$F_Q(Z_1, Z_2, n) = \frac{P_1(n)P_2(n) - Q_1(n)Q_2(n)}{R_1(n)R_2(n) - S_1(n)S_2(n)} \,.$$

A specific example to illustrate might be

$$F_Q(\{P, Q, R, S\}, \{P, Q, S, R\}, n) = \frac{P(n)P(n) - Q(n)Q(n)}{R(n)S(n) - S(n)R(n)} \,.$$

Thus one can form a fraction from any two permutations of Z. Similarly, higher order fractions can be defined. This is left for future papers.

A number of interesting identities similar to those of the linear fractions section have been found for the quadratic case. Further details are left for future papers.

6 Discussion

An introduction to Ramamurthi fraction forms has been given. A number of definitions have been given, and some introductory fraction results have been shown. Basic techniques for manipulating these forms were presented.

These results are related to the Ramanujan TB formula, and this specific motivation is discussed in this paper. More specific relationships to the Ramanujan TB formula have been stipulated.

A brief motivation for higher order forms has been introducted. More results will appear in future papers.